

# Newton's law in an effective non commutative space-time

**Alessandro Gruppuso**

Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna  
via Irnerio 46, 40126, Bologna, Italy

and

†INAF/IASF, Istituto di Astrofisica Spaziale e Fisica Cosmica

Sezione di Bologna

via Gobetti 101, 40129, Bologna, Italy

E-mail: [gruppuso@bo.iasf.cnr.it](mailto:gruppuso@bo.iasf.cnr.it)

## **Abstract.**

The Newtonian Potential is computed exactly in a theory that is fundamentally Non Commutative in the space-time coordinates. When the dispersion for the distribution of the source is minimal (i.e. it is equal to the non commutative parameter  $\theta$ ), the behavior for large and small distances is analyzed.

PACS number: 11.10.Nx

† Present address.

*Introduction.* Recently a new model of Quantum Field Theory on Non-Commutative space time, satisfying Lorentz Invariance and Unitarity, has been proposed in [1, 2]: it is shown that there is no need of Weyl-Wigner-Moyal  $\star$ -product [3, 4, 5] and the Non-Commutativity is carried by a gaussian cut-off in the Fourier Transform of the fields. This is not an ad hoc regularization device but is a result coming from the averaging operation on coherent states. This cut-off (which depends on the Non-Commutative parameter  $\theta$  [2]) is also present in the (Feynman) Propagator and is responsible for the UV finiteness of the theory [1, 2, 6].

The aim of this paper is to compute the modification of the Newtonian Potential due to the change of the Green's Function caused by the Non-Commutativity of the space-time coordinates.

In order to deal with a theory that is effectively Non-Commutative in the space-time coordinates, we follow [1] and suppose that the propagator satisfies:

$$-\square_x G(x, x') = \left(\sqrt{\frac{a}{\pi}}\right)^4 e^{-a(x-x')^2}, \quad (1)$$

where  $\square_x = \partial_t^2 + \nabla_x^2$ ,  $a = 1/4\theta$  and  $\theta$  is the Non-Commutative parameter with the dimensions of an area  $\ddagger$ . Notice that in the limit  $\theta \rightarrow 0$  the gaussian distribution becomes a delta function and standard commutative theory is recovered. As usual the Newtonian Potential  $V(x)$  is related to the fluctuation of the 00 component of the metric  $h_{00}(x)$  through  $V(x) = h_{00}(x)/2$  and the fluctuation is given by [7]:

$$h_{00}(x) = -8\pi G_N \int d^4x' G(x, x') T_{00}(x'), \quad (2)$$

where  $G_N$  is the Newton's constant and  $T_{00}$  is the 00 component of the stress-tensor of the source of the gravitational field. Since in a theory that is fundamentally Non-Commutative the concept of point is meaningless, the source cannot be a delta function but will be given by a gaussian distribution:

$$T_{00}(x) = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} f(q^2), \quad (3)$$

where  $f(q^2) = e^{-\alpha \vec{q}^2} M$ , with  $\alpha$  and  $M$  constants. We shall choose  $\alpha = \theta$  because we want to consider a minimum dispersion for the source.

The Fourier Transform of the propagator is called  $G(k)$  and it is implicitly defined as

$$G(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} G(k). \quad (4)$$

$\ddagger$  We are considering an euclidean signature of the metric, coming from a Wick rotation of the time coordinate.

Considering that the Fourier Transform of a gaussian distribution is still a gaussian function,

$$\left(\sqrt{\frac{1}{4\theta\pi}}\right)^4 e^{-\frac{1}{4\theta}(x-x')^2} = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x-x')^\mu} e^{-\theta(k_0^2 + \vec{k}^2)}, \quad (5)$$

then  $G(k)$  can be obtained through equation (1):

$$G(k) = \frac{e^{-\theta(k_0^2 + \vec{k}^2)}}{k_0^2 + \vec{k}^2}. \quad (6)$$

From equations (2),(3),(4) and (6) one can write

$$h_{00}(x) = -8\pi G_N \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \frac{e^{-\theta(k_0^2 + \vec{k}^2)}}{k_0^2 + \vec{k}^2} \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{x}'} f(q^2). \quad (7)$$

Since the source does not depend on time we can use

$$\int dt' e^{-ik_0 t'} = 2\pi \delta(k_0), \quad (8)$$

and with the help of

$$\int d^3x' e^{i\vec{x}' \cdot (\vec{k} - \vec{q})} = (2\pi)^3 \delta(\vec{k} - \vec{q}), \quad (9)$$

it is possible to simplify (7), obtaining the following equation:

$$h_{00}(x) = -8\pi G_N \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \frac{e^{-\theta \vec{k}^2}}{\vec{k}^2} f(k^2). \quad (10)$$

Performing the integration over the angular part of  $d^3k$  one finds

$$h_{00}(r) = -\frac{2G_N M}{\pi} \frac{1}{ir} \int_{-\infty}^{\infty} dk k e^{ikr} \frac{e^{-(\theta + \alpha)k^2}}{k^2 + \epsilon^2}, \quad (11)$$

where the definition of  $f(k^2)$  has been used, with  $k = \sqrt{\vec{k}^2}$ , and the limit  $\epsilon \rightarrow 0$  is understood §. The integration can be performed exactly [9] and the Newtonian Potential is found to be the following:

$$V(r) = -G_N \frac{M}{r} \text{Erf} \left( \frac{r}{2\sqrt{\theta + \alpha}} \right), \quad (12)$$

where  $V(r) = h_{00}(r)/2$  has been used.

First of all we notice that if  $\alpha \gg \theta$  then the Non Commutativity is screened by the source. This is natural since the source must be localized as much as possible in space. But in a theory that is fundamentally Non-Commutative, the smallest spread is given by  $\theta$ .

§ As usual this limit has been introduced in order to regularize the integrand at  $k^2 = 0$ .

We set now  $\alpha = \theta$  since we wish to analyze the minimum case. In the region for which  $r \gg 2\sqrt{2\theta}$  one obtains:

$$V(r) = -G_N \frac{M}{r} \left[ 1 + e^{-r^2/(8\theta)} \left( -\frac{2}{\sqrt{\pi}} \frac{\sqrt{2\theta}}{r} + \mathcal{O}\left(\frac{2\sqrt{2\theta}}{r}\right)^3 \right) \right]. \quad (13)$$

Numerically (on requiring that the correction is of order 1) one finds that the large distance correction becomes important for the following critical value

$$\frac{r_c}{2\sqrt{2\theta}} \simeq 0.4576. \quad (14)$$

Since it is known that Newton's Law is verified up to a distance of the order of  $200 \mu\text{m}$  [8], it is then possible to find a constraint for  $\theta$ ,  $\theta < 10^{-8} m^2$ . Of course due to the fact that Newton's Law is not tested for very small distances, this constraint is not so strong. One has to use precision measurements in order to have a significant bound. In the opposite region where  $r \ll 2\sqrt{2\theta}$  one obtains

$$V(r) = -G_N \frac{M}{r} \left[ \frac{r}{\sqrt{\pi(2\theta)}} + \mathcal{O}\left(\frac{r}{2\sqrt{2\theta}}\right)^3 \right]. \quad (15)$$

It is interesting to note that there is no divergence (unlike the Commutative case): the introduction of the  $\theta$  parameter gives a minimal length that regularizes the theory:

$$V(0) = -G_N \frac{M}{\sqrt{\pi(2\theta)}}. \quad (16)$$

*Conclusions.* The main result of this brief paper is given by equation (12). The calculation is exact and no approximation has been made. We notice that there is a deviation from the standard Newtonian Potential but the shape of the source can screen the effect of Non-Commutative space. In the minimal case (i.e. the dispersion for the profile of the source is given by  $\theta$ ) such a deviation has been computed for the large and small distance approximations. In the first case we used the corrections to Newton's Law in order to constraint the Non-Commutative parameter (but the obtained bound is weak since Newton's Law is verified down to distances of the order of  $200 \mu\text{m}$ ). In the second case it is seen that there is a regular behavior at  $r = 0$ . This reflects the well known fact that Non-Commutativity introduces a minimal length which regularizes the theory.

*Aknoledgements.* It is a pleasure to thank Euro Spallucci for useful suggestions and many comments on the draft version of this paper. I am also

grateful to Piero Nicolini, Anais Smailagic and Giovanni Venturi for interesting discussions. I thank Roberto Casadio for collaboration in the early stage of this work.

## References

- [1] A.Smailagic and E.Spallucci, *UV divergence-free QFT on noncommutative plane*, Published in J.Phys.A36:L517-L521,2003; hep-th/0308193.
- [2] A.Smailagic and E.Spallucci, *Lorentz invariance, unitarity in UV-finite of QFT on Noncommutative spacetime*, Published in J.Phys.A37:1-10,2004, Erratum-ibid.A37:7169,2004; hep-th/0406174.
- [3] H.Weyl, *Quantum Mechanics and Group Theory* (In German), Published in Z.Phys.46:1,1927.
- [4] E.P.Wigner, *On The Quantum Correction for Thermodynamic Equilibrium*. Published in Phys.Rev.40:749-760,1932.
- [5] J.E.Moyal, *Quantum Mechanics as a Statistical Theory*. Published in Proc.Cambridge Phil.Soc.45:99-124,1949.
- [6] P.Nicolini, *Vacuum Energy Momentum Tensor in (2+1) NC Scalar Field Theory*, hep-th/0401204.
- [7] R.M. Wald, *General relativity*, Chicago University Press, Chicago (1984).
- [8] C.D.Hoyle, U.Schmidt, B.R.Heckel, E.G.Adelberger, J.H.Gundlach, D.J.Kapner, H.E.Swanson, *Submillimeter tests of gravitational Inverse Square Law: a Search for 'Large' Extra Dimensions*. Published in Phys.Rev.Lett.86:1418-1421,2001; hep-ph/0011014.
- [9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press,(1965).